Black Holes and How to Study Them

Analogue models of Gravity

Joseph Balsells

ASTR 320 December 2017

The earliest Greek philosophers called the world as a whole the *kosmos*. This noun derives from a verb which meant 'to order', 'to arrange', 'to marshall' – it was used by Homer of the Greek generals marshaling their troops for battle [1]. Thus the Greek philosophers believed the universe to have an order. But they believed in something more subtle too. The word *kosmos* also had a connotation of beauty which is carried today in the word cosmetics [1]. That is, the Greeks thought of the order apparent in nature as being beautiful. Modern astrophysics deals with strings, loops, stars, galaxies, black holes, gravitational waves, dark matter, dark energy, the cosmic microwave background, and the origin and fate of the universe itself. As it was then, so also now it is the goal of science to bring order to these initially discordant systems. The unraveling of this theory requires the interplay between all subfields of physics. In studying the very smallest things we use quantum field theory. In studying the very largest we rely on cosmology. It is the goal of physicists to develop these individual studies to the fullest extent so that no individual system is left unexplained. The pursuit of astrophysics is in tying together this work into a self-consistent theory of the universe.

Traditionally, observation and theoretical argument have been the important steps to developing theory. In the scientific method the theories are the end-product while observation is both their genesis and validation. In the early days this process consisted mainly of theoretical argument and relatively little observation. This is not explicitly a derogatory statement. Anaximander (610-540 BC) believed the universal principle of all things to be 'the infinite' [1]. Infinite in extent and also in characteristics. He posited that all things derived from this 'infinite.' Anaximenes (585-528 BC) countered Anaximander by positing that the universal principle of all things was actually infinite air [1]. And so it was for a time. Observation has not always held the place it did when Galileo looked through the first telescope and saw the moons of Jupiter.

It has long been recognized that physics is entering a stage not unlike that of the Greeks. Physics is becoming detached from its empirical foundations. Never again will we be as naive as the physicists of a century ago to claim that our theories are complete. Nonetheless it is true that many modern theories no longer make testable predictions. Observation and speculation have become untethered once more. Just as the Greeks spoke of infinite air being the principle of all things, so now scientists speak of strings and loops as the principle of all things. One can hardly fault them. Experiments in modern physics are hard. Gravity is the last force to be fully understood in part because it seems intractable to experiment with. Manipulating even weak gravitational fields requires manipulating masses on the order of planets. What can be done to reconnect physics to the empirical world?

Today there are many signs pointing at black holes as centers for groundbreaking physics. Black holes combine in some yet unknown way the quantum theory of the small with gravitation, relativity, and thermodynamics. This connection carries weighty implications for foundational problems in the quantum theory of entanglement and information theory. Black holes cannot be studied under laboratory settings. Black holes are barely observable. The fact that we cannot study Hawking radiation experimentally puts a halt to the ordinary scientific method. Something has to be done if any progress is to be made to constrain theory.

Purely theoretical arguments are usually linked to observation and validation by a hier-

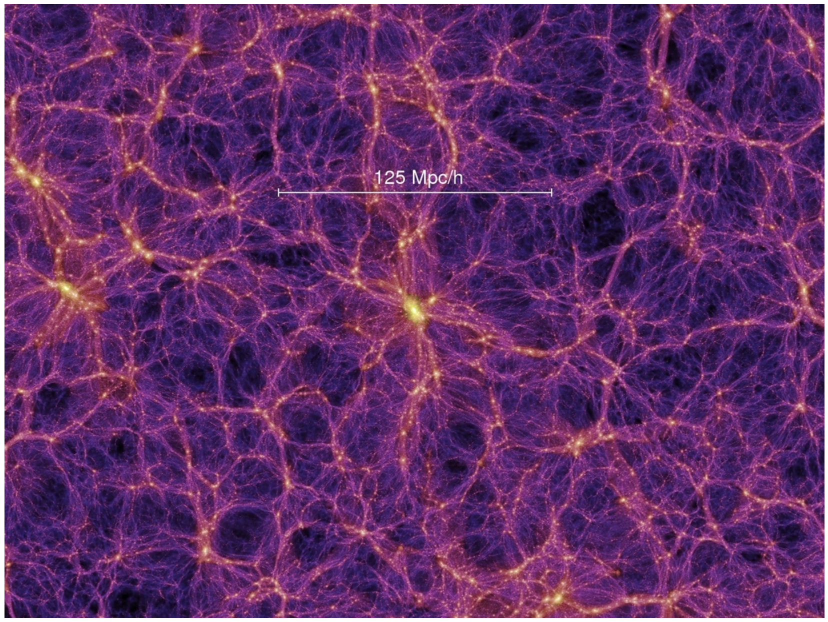

Figure 1: Snapshot of the Millennium Simulation, a cosmological simulation of 10 billion particles over a region of the universe 2 billion light-years on a side. Springel, V., et al. Simulating the joint evolution of quasars, galaxies and their large-scale distribution. *Nature.* 2005.

archy of models. The hope is that, if the models are any good, they will simplify the analysis of the system at hand. Some models are purely analytic, consisting of a series of equations, others are numerical, living out their lives in computer code. An example of an analytic model is the Friedmann-Robertson-Walker cosmology for an isotropic, expanding universe. An example of a numerical model is the Millennium simulation in Figure 1. Analytical models tend to associate more closely with the purely theoretical arguments while numerical models tend to associate more readily with observations. In some sense analytic models *are* the theories and numerical models act *as sources for* observation. We have models of both kinds to study black holes. Numerical simulations are very interesting but mostly outside of the scope of this paper. A review of attempts to simulate general relativity in the weak-field regime is given in [2]. The analytical models concern solutions to Einstein's field equations. The most general solution to Einstein's Field equations for black holes are the Kerr-Newman Family.

Kerr-Newman black holes are axis-symmetric with total mass-energy M, angular momentum $J = |\vec{J}|$, and electric charge e. A series of proofs given in the late 1960s by Werner Israel and others suggest that these are the only three measurable quantities for a black hole. The proofs are called no-hair theorems in reference to the idea that black holes are described by only three numbers and "have no hair." This is remarkable and makes no sense. Black holes start out as stars with many degrees of freedom. Stars contain temperature gradients, density perturbations, electromagnetic fields and an infinitude of particles. Upon gravitational collapse all these microstates are erased in favor of just three parameters. This was the hint that lead Jacob Bekenstein in 1972 to realize that black holes must have very high entropy [3]. And thus black hole thermodynamics was born.

To understand the context of black hole thermodynamics it is useful to understand the difficulty of applying thermodynamics to any gravitational system. In class we covered the basic thermodynamic quantities of a self-gravitating system. There we considered a self-gravitating system of N particles in virial equilibrium i.e., N particles interacting via Newtonian gravity such that the total energy of the system is given by $2K + W = 0$, where K is the kinetic energy of all the particles and W is their gravitational potential energy. We found that we may approximate this system's energy as [4]

$$E = -K = -\frac{3}{2} N k_b T$$

From this it follows that the heat capacity of such a system is

$$C \equiv \frac{dE}{dT} = -\frac{3}{2} N k_b$$

Contrary to standard intuition, this implies that by losing energy a gravitational system gets hotter. The implication of this is that gravitational systems are inherently unstable. If a self-gravitating system is in contact with another system at lower temperature it will give up energy to the colder system. Ordinarily this brings the temperature of the system down and the other up so that they meet in the middle. However gravitational systems get hotter which only increases their propensity to shed energy to cooler systems. This points to the fundamentally difficulty in treating the thermodynamics of self-gravitating systems: they lack true equilibria and may only be said to be metastable. The gravitational collapse of dust and gas clouds formed the first galaxies. Within these galaxies the dust continued to collapse under its own weight until it was dense enough for the first stars to be born. As stars live they stabilize themselves against further collapse for only a finite time before even they collapse under their own weight. For stars greater than about 5 solar masses it is only this final end stage as a black hole that appears to be stable. In the end, the full understanding of all the systems of astrophysics will require an exquisite understanding of gravity on all scales, from the Universe to the singularity.

Newton gave the first modern description of gravity in his *Philosophiae Naturalis Principia Mathematica* in 1687. In 1796 Pierre-Simon de Laplace gave the first[1] description of a body so massive that not even light could escape its surface [5]. Using only Newtonian theory Laplace was able to deduce the correct radius for a body to be dense enough so that its gravity became so strong that even light could not escape. His result was that the radius should be equal to $2GM/c^2$. This radius is about 1km for the Sun and 1cm for the Earth. It is also known as the Schwarzschild radius, after Karl Schwarzschild who derived it in 1916 as the solution to Einstein's field equations for a point mass. Using this result Laplace conjectured, "The largest bodies in the universe may thus be invisible by reason of their magnitude" [5]. It is remarkable how well Newtonian theory extends to describe systems.

Despite its wide success however, there are numerous cases for which Newton's formulation is incomplete. The clearest case of Newton's theory giving the wrong result pertains to measurements of the precession of the perihelion of Mercury. Mercury's orbit is highly eccentric giving it a perihelion (nearest point to the sun) which is nearly half its aphelion (furthest point from the sun). This makes measuring the location of the perihelion of Mercury relatively easy and so it had been known for some time that Mercury's perihelion

[1]Actually John Michell gave a similar geometric argument in 1784 but it was Laplace who provided a mathematical treatment [5].

precesses around the Sun. Newtonian theory actually predicts that the location of the perihelion should precess but it gives the wrong number. Only general relativity, published by Einstein in 1915, gives the correct answer. General relativity also provides much more powerful results for what might happen at the event horizon for a black hole. The important results of Einstein's theory are summarized in two sets of equations:

$$\frac{d^2x^\mu}{ds^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \tag{1}$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2}$$

The first of these is called the geodesic equation and the second are known as the Einstein field equations. The geodesic equation is the appropriate spacetime analog of Newton's $F = Ma$. The geodesic equation gives the motion of a particle in a curved spacetime. On its l.h.s. the second derivative of the 4-position (an acceleration) appears. The r.h.s describes the curvature of spacetime. The right-most terms give the directions of movement while the Christoffel symbol, $\Gamma^\mu_{\alpha\beta}$, accounts for spacetime curvature. The geodesic equation says that particles follow straight paths in curved spacetimes (a straight path in a curved space is called a geodesic, hence the name). The Christoffel symbol is a combination of the metric tensor, $g_{\mu\nu}$. The value of the metric tensor is solved for using the Einstein field equations. The key variables in the Einstein field equations (2) are the metric tensor $g_{\mu\nu}$ and the energy-momentum tensor $T_{\mu\nu}$. The energy-momentum tensor represents the spatial distribution of energy-momentum and the metric tensor $g_{\mu\nu}$ is a measure of the geometry of spacetime. Loosely, $g_{\mu\nu}$ gives the distance between any two points on a possible curved space.[2] The other terms, $R_{\mu\nu}$ and R are also combinations of the metric tensor and Λ is an

[2] The ordinary distance between two points is

$$dl^2 = dx^2 + dy^2 + dz^2$$

ordinary constant. Hence the Einstein field equations can be seen as an equation for $g_{\mu\nu}$. The following mnemonic is useful: energy-momentum tells space how to curve, and the geodesic equation tells energy-matter how to move in the curved space. Einstein's field equations also neatly account for Maxwell's laws of electromagnetism. Maxwell's laws are fundamentally relativistic and are easily formulated in a way which respects spacetime curvature. Together equations (1) & (2) constitute general relativity. So the theory of optics fits nicely into general relativity, but also optics can account for many features of general relativity.

In fact, the earliest legitimate analogies between gravity and optics came from Pierre de Fermat, an early contemporary of Newton. Fermat's principle of least time states that light rays move along paths which minimize their transit time. Consider the path traveled by light in a medium with a position-dependent index of refraction, $n(\vec{x})$. The index of refraction of a medium is defined as the ratio between the vacuum speed of light and its speed through that medium:

$$n(\vec{x}) = \frac{c}{v(\vec{x})}$$

More generally in a curved space the dependence on the coordinates x, y, z could be more general,

$$dl^2 = \alpha dx^2 + \beta dy^2 + \gamma dz^2$$

Where α, β, γ could depend on location. In fact the most general possible description of distance in three dimensions could include cross terms like $dxdy$ or $dydz$. Therefore in full generality we have

$$\begin{aligned} dl^2 = {} & g_{11}dx^2 + g_{12}dxdy + g_{13}dxdz \\ & + g_{21}dydx + g_{22}dy^2 + g_{23}dydz \\ & + g_{31}dzdx + g_{32}dzdy + g_{33}dz^2 \end{aligned}$$

This is written more succinctly in matrix (tensor) form as

$$dl^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}dx^i dx^j$$

Where we identify (x^1, x^2, x^3) with (x, y, z). In this way we see that the ordinary definition of distance is merely a special case of this more general form with $g_{ij} = \delta_{ij}$. So the metric tensor gives a description of the geometry of the space.

In particular the speed of light in a medium is always less than that in a vacuum. The time it takes to travel along a path γ parameterized by s is then given by

$$T_\gamma = \int_\gamma \frac{n(\vec{x}(s))}{c} ds$$

Since c is constant we may pull it out of the integral and we are left with a integral of the distance along the path weighted by the index of refraction. Fermat's principle is then equivalent to minimizing this "optical distance"

$$L_\gamma = \int_\gamma n(\vec{x}(s)) ds$$

Restating the optics problem, the goal is to determine the path of shortest length in a space where distance at each point is not equal. That is, Fermat's principle is equivalent to searching for the geodesics of a curved space. It isn't hard to see that the appropriate metric in this case is

$$g_{ij} = n(\vec{x})^2 \delta_{ij}$$

This connection between optics and general relativity is nontrivial and sometimes a powerful tool in studying one field or the other. For example, in general relativity one can write the Schwarzschild metric in a basis such that it takes an optical form with the spherically symmetric form [6]

$$n(r) \approx 1 + \frac{2M}{r}$$

This motivates the search for further analogies between general relativity and optics. The key feature in getting this program to work is control over the form of $n(\vec{x})$ i.e., asserting fine control over the optical properties of various condensed-matter systems. With such control many spacetime geometries become laboratory testable. In 2010 Qiang Cheng and colleagues in China developed a black hole analogue for light in the microwave regime using condensed-matter metamaterials [7]. Even more dramatically, in 2009 Igor Smolyaninov at the University of Maryland – College Park developed a condensed-matter theory for a metamaterial whose optical properties simulate the geometry of a spacetime with 2 time and 2 spatial dimensions [8]. Suprisingly, Smolyaninov finds an that this material can undergo a phase transition reminiscent of the big bang. Specifically, he finds that the (2+2) spacetime geometry can transition into a (2+1) Minkowski spacetime together with a large number of particles being generated during the transition [8]. This sort of behavior is predicted in string theory. Before the big bang, string theory predicts 11 dimensions which become furled up during the bang. As they furl all of the energy modes which oscillated in these extra dimensions gets pushed into our regular three as the material which later formed us. Metamaterials are only one avenue toward simulating gravity in analogue systems.

Returning to black holes specifically, in 1974 Stephen Hawking made the first connection between black holes and quantum theory. Bekenstein's 1973 work [4] previously predicted black holes to have a temperature proportional to their surface gravity, κ. Bekenstein's result was based on considering black holes as perfect black bodies – they absorb all incident radiation. Using quantum field theory (and independently of Einstein's field equations) Stephen Hawking showed that in addition to absorbing, black holes should also radiate particles. Via this mechanism he showed that black holes with a mass less than $10^{15}g$ created early in the age of the universe would have evaporated completely by now. He calculates that in the last 0.1 seconds a black hole should radiate 10^{30} erg before completely evaporating. This is equivalent to 1 million 1 Mton hydrogen bombs and should be observable [9]. On the basis of his calculation Hawking also determined the proportionality factors relating the temperature to the surface gravity to give

$$T = \frac{\kappa}{2\pi}\frac{\hbar}{2k_b}$$

It is hard to overstate the importance of this result. For a Schwarzschild black hole we have

$$\kappa = \frac{c^3}{4GM}$$

Thus, substituting,

$$T_{schwarzschild} = \frac{c^3 \hbar}{16\pi G M k_b}$$

This is one equation relating the fundamental constants of gravity, G, quantum mechanics, $\hbar$, relativity, c, and thermodynamics, k_b. Hawking radiation is a big sign that something very fundamental is occurring around the event horizon of a black hole. Black holes themselves were only detected relatively recently and the hope for experimentally measuring Hawking radiation from a gravitational black hole is slim. Because our chances of validating Hawking's prediction directly are small theorists worked to make the theory as airtight as possible.

In 1981 William Unruh recognized [11] a potential flaw in Hawking's argument. Hawking's calculations depended on the behavior of quantum field theory at wavelengths below the Planck length where we know quantum field theory must break down. This extrapolation is essentially unjustifiable. Theodore Jacobson gave a proof in 1991 in [12] that the spectrum may be cut off at the Planck length without affecting the qualitative behavior, but Unruh had already suggested another approach. He showed that since Hawking's derivation was independent of the Einstein field equations, it was actually independent of general relativity i.e., that Hawking radiation is not a phenomenon limited to black holes. Unruh then presented an analog system which should also exhibit Hawking radiation. Unruh's idea was to consider sound in a flowing fluid. In particular he chose a non-relativistic, irrotational, barotropic fluid where some regions of the flow were supersonic. A realization of this flow is given for example by a Laval nozzle as in Figure 2. Such a nozzle can create a boundary in a fluid where the flow transitions from subsonic to supersonic. Sound waves created in the

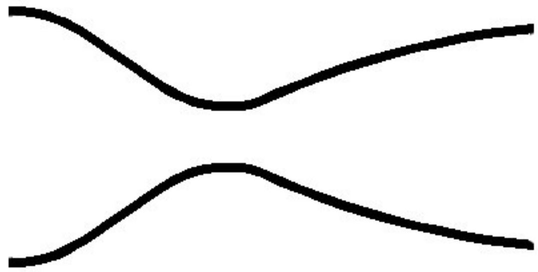

Figure 2: Basic schematic of the Laval nozzle. Flow enters from the left, is constricted in the throat, and exits to the right. When the flow entering is subsonic the constriction can cause the flow to become supersonic. By turning the nozzle the other way supersonic flow can be made to become subsonic.

supersonic region then would never be able to propagate back to the subsonic region. Here the event horizon consists of the surface in the throat of the nozzle where the flow transitions from subsonic to supersonic. Unruh's realization was that sound waves, or when viewed as particles phonons, created near the analogue horizon would share the same features as real Hawking radiation.

Deriving this result requires little more than what we have already done in class. Since the flow is irrotational we have,

$$\nabla \times \vec{u} = 0 \implies \vec{u} = \nabla\psi$$

and because the fluid is barotropic, $P = P(\rho)$. To describe the fluid system we begin with the ordinary continuity and Euler equations (without viscosity)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \qquad \text{(Continuity)}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla P}{\rho} \qquad \text{(Euler)}$$

and assume that ρ_0, P_0, and ψ_0 are exact solutions to a stable background flow. We then introduce perturbations

$$\rho(t,x) = \rho_0(t,x) + \rho_1(t,x)$$

$$P(t,x) = P_0(t,x) + P_1(t,x)$$

$$\psi(t,x) = \psi_0(t,x) + \psi_1(t,x)$$

As we showed in class we can substitute these back into the Euler and continuity equations to find equations for the perturbations ρ_1, P_1, ψ_1. Doing so we obtain

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_1 \nabla \psi_0 + \rho_0 \nabla \psi_1) = 0$$

$$\rho_0 \Big(\frac{\partial \psi_1}{\partial t} + \nabla \psi_0 \cdot \nabla \psi_1 \Big) = P_1$$

$$P_1 = c_{\text{sound}}^2 \rho_1$$

The first two equations give the evolution of ρ_1 and ψ_1 while the third says that P_1 can be obtained from ρ_1 as should be the case in a barotropic fluid. Now the two first order equations can be combined into a single second order equation containing the same information by taking derivatives and substituting. The result is [6]

$$\frac{\partial}{\partial t} \Big(c_{\text{sound}}^{-2} \rho_0 \Big(\frac{\psi_1}{\partial t} + \vec{v}_0 \cdot \nabla \psi_1 \Big) \Big) = \nabla \cdot \Big(\rho_0 \nabla \psi_1 - c_{\text{sound}}^{-2} \rho_0 \vec{v}_0 \Big(\frac{\partial \psi_1}{\partial t} + \vec{v}_0 \cdot \nabla \psi_1 \Big) \Big)$$

Now this is supposed to be easily recognizable[3] as the propagation of scalar waves in a

[3] It wasn't to me

black hole spacetime i.e., this equation is identical to the following

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^{\nu}}\psi_1\right) = 0$$

With the metric given by

$$g^{\mu\nu}(t,\vec{x}) \equiv \frac{1}{\rho_0 c_{\text{sound}}}\begin{bmatrix} -1 & \vdots & -v_0^j \\ \cdots\cdots & . & \cdots\cdots \\ -v_0^i & \vdots & (c_{\text{sound}}\delta^{ij} - v_0^i v_0^j \end{bmatrix}$$

Understanding this last step is not the important point here. The point is that the regular fluid flow problem has been mapped exactly onto a general relativistic problem. Unruh continued to then provide explicit derivation of the equivalent Hawking radiation temperature in this problem. The particles created in the hydrodynamic system are of course sound waves, but the particle description is to call them phonons. He indicates that on either side of the surface where the flow is supersonic, phonons could be created which would mimic Hawking radiation. The predicted temperature of this phonon Hawking radiation is however very small and ordinary water is much to warm to observe this process. So recently interest has been directed to following this analogy into cold fluids. Specifically, work has been done with Bose-Einstein condensates and superfluids like helium-4 [14]. Jeff Steinhauer claims to have measured this analog Hawking radiation in a Bose-Einstein condensate [15]. Although skepticism surrounds this claim, even if it is not true it will be sooner or later. And this will give us an experimental grasp on Hawking radiation.

One hope of the fluid approach to simulating gravity is that, unlike gravity, the full microscopic description of fluid dynamics is known. That is, as fluid dynamics is only a continuum approximation for physical fluids, so is field theory in curved spacetime only an approximation to quantum gravity. If we can relate the two continuum approximations to each other, hopefully the microscopic theory of fluids can give hints to the microscopic theory of gravity. It is too early now to know precisely what can be gleaned from fluid analogues.

We have seen two different paradigms for simulating gravity in analogue systems. The first was the based on condensed-matter physics and metamaterials. By asserting control over the dielectric and magnetic properties of these materials researchers have created artificial spacetimes. The other model, the superfluids and Bose-Einstein condensates, gives a laboratory hold on Hawking radiation and other black hole phenomena. These systems are still new and it is yet unclear exactly how useful they will be. If the past is any indication as to what we can expect from them, it is that we will be should be excited for a world of new surprises. Greek philosophy began in 585 BC when Thales of Miletus predicted an eclipse of the Sun. One can only wonder at what will come after Smolyaninov's model of the big bang.

REFERENCES

[1] Barnes, J. *Early Greek Philosophy*, Penguin Books (1987).

[2] Adamek, J., D. Daverio, R. Durrer, M. Kunz, General Relativistic N -body simulations in the weak field limit, *Phys. Rev. D* **88**, 103527 (2013).

[3] Bekenstein, J. D. Black holes and entropy. Phys. Rev. D 7, 23332346 (1973).

[4] van den Bosch, F. Physical Processes in Astronomy, unpublished notes available at https://campuspress.yale.edu/astro320/

[5] Montgomery, C., W. Orchiston, I. Whittingham, Michell, Laplace and the origin of the black hole concept, *J. Astron. Hist. Herit.*, **12(2)**, 90-96 (2009).

[6] Novello, M. , M. Visser, G. Volovik, *Artificial Black Holes*, World Scientific (2002).

[7] Cheng, Q., T. Cui, W. Jiang, B. Cai, An electromagnetic black hole made of metamaterials, arXiv:0910.2159 [physics.optics]

[8] Smolyaninov, I. Optical models of the big bang and non-trivial space-time metrics based on metamaterials, arXiv:0908.2407 [physics.optics]

[9] Hawking, S. W. Black hole explosions?, *Nature* **248** (1974) 30-31.

[10] Davies, P. C. W. Thermodynamics of Black Holes, *Rep. Prog. Phys.* **41**, 1313 (1978).

[11] Unruh, W.G. Experimental black hole evaporation?, *Phys. Rev. Lett.* **46**, 1351 (1981)

[12] Jacobson, T. Black hole evaporation and ultrashort distances, *Phys. Rev. D* **44**, 1731 (1991).

[13] Unruh, W.G. Dumb holes and the effects of high frequencies on black hole evaporation, *Phys. Rev. D* **51**, 2827 (1995) [gr-qc/9409008]. (Title changed in journal: "Sonic analog of black holes and...")

[14] Hennigar & Mann, Superfluid Black Holes, *Phys. Rev. Lett.* **118** (2017).

[15] Steinhauer. J. Observation of Quantum Hawking radiation and its entanglement in an analogue black hole, *Nature Physics* **12**, 959965 (2016).

[16] Springel, V., et al., Simulating the joint evolution of quasars, galaxies and their large-scale distribution, *Nature* **435**, 629636 (02 June 2005)