Dijkgraaf-Witten Theory

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Contents

Outline

The goal of this paper is to introduce the reader to Dijkgraaf-Witten theory. Dijkgraaf-Witten theory is a topological quantum field theory. Topological quantum field theories are today rewarding area of study in both physics and mathematics at the intersection of physics, geometry, and algebra. Their usefulness stems mainly from the definite axiomatization given by Atiyah naturally expressed in the language of category theory as symmetric monoidal functors from the category of bordisms to the category of vector spaces. Instead of starting immediately with this definition and because it is not assumed that the reader has any background with quantum field theory, we provide some motivation through developing the path integral of quantum field theory. Nor have we assumed any familiarity with category theory and the necessary definitions are developed in the course of the exposition. The outline of the paper is as follows: The first section provides a short introduction to quantum field theory and the philosophy of topological quantum field theory. We then state Atiyah's axioms for a topological quantum field theory and expound upon its structures for dimension 2. Finally we discuss Dijkgraaf-Witten theory and provide connections to classical representation theory via Mednykh's formula.

Quantum Mechanics

Because quantum field theory is the relativistic successor to quantum mechanics and because many of the interpretations of TQFTs depend on objects first encountered in quantum mechanics we begin our development with quantum mechanics.

In classical mechanics the state of a particle is fully determined by its position x and momentum p. Together these coordinates define a point in a 6-dimensional phase space and the evolution of the particle is given as a trajectory through this phase space. In quantum mechanics the state of a particle is given by a vector $|\psi\rangle$ living in a complex Hilbert space $\mathcal{H}$. We denote the dual vector as $\langle\psi| \in \mathrm{Hom}(\mathcal{H}, \mathbb{C})$. Thus $\langle\psi|\psi\rangle \in \mathbb{C}$ is convenient notation for an inner product. Max Born gave the statistical interpretation of $|\psi\rangle$ which says that if $\mathcal{H}$ has a basis, say $|x\rangle$, then the projection of the state vector onto this basis is related to the probability of observing the particle in the state $|x\rangle$. The projection is called the wavefunction ψ and is in general complex-valued

$$\psi(x) \equiv \langle x|\psi\rangle$$

and $|\psi(x)|^2$ is the probability density of observing the particle at x. This interpretation requires that ψ is properly normalized, or that $\langle\psi|\psi\rangle = 1$.

The state vector will in general depend on time. This is indicated by writing $|\psi(t)\rangle$. When the state vector is viewed as describing the position of a particle it can be projected onto the position basis giving the time-dependent wavefunction, $\Psi(t, x) = \langle x|\psi(t)\rangle$. According to Born's interpretation $|\Psi(t, x)|^2$ gives the probability density of finding the particle at position x at time t. It should be remarked that the state vector is more fundamental and general than this position representation leads one to believe. By analogy with classical mechanics we expect that the wavefunction can also be projected onto the momentum-space basis. In fact this is a valid operation and when this is done it is sometimes referred to as $\Phi(t, p) \equiv \langle p|\psi(t)\rangle$. Unlike in classical mechanics however position and momentum are not totally independent. In quantum mechanics the position and momentum representations are related roughly by Fourier transformation which leads to the well-known Heisenberg uncertainty principle. Also unlike classical mechanics these representations are not always sufficient, the Hilbert space $\mathcal{H}$ need not be related to either of these representations in general.

The goal of quantum mechanics is to determine how states evolve in time and for this we need the idea of an operator acting on the state. We define operators in $\mathcal{H}$ as assignments $\hat{O} : \mathcal{H} \to \mathcal{H}$ and similarly in the dual space. Every operator $\hat{O}$ acting in the Hilbert space can be identified with an operator in the dual space thus defining its adjoint $\hat{O}^\dagger$. The fundamental goal of quantum theory is to determine how the state vector evolves in time. We may postulate that the time-evolution of $|\psi(t)\rangle$ is given by an operator U as $|\psi(t)\rangle = U(t)|\psi_0\rangle$ where $|\psi_0\rangle = |\psi(0)\rangle$ for some chosen origin of time. Note that $U(0) = 1$ is the identity operator. Using this operator we require consistency with Born's interpretation that total probability is conserved

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi_0|U^\dagger(t)U(t)|\psi_0\rangle = 1$$

Thus we must have $U^\dagger U = 1$. This condition says that U is *unitary*. For small t we expect that U should remain close to the identity, $U(\epsilon) \approx 1 - i\epsilon H$ where H is some other operator and the imaginary number i is introduced for later convenience but might as well have been included in the definition of H. This condition expresses the belief that systems evolve continuously. Because of unitarity this condition becomes

$$U^\dagger U = (1 + i\epsilon H^\dagger)(1 - i\epsilon H) = 1 + i\epsilon(H^\dagger - H)$$

where we have kept only terms to first order in ϵ. To ensure unitarity of U this condition requires $H = H^\dagger$, or that H is hermitean. Thus we have

$$|\psi(t+\epsilon)\rangle = (1 - i\epsilon H)|\psi(t)\rangle$$

$$\implies \frac{|\psi(t+\epsilon)\rangle - |\psi(t)\rangle}{\epsilon} = -iH|\psi(t)\rangle$$

In the limit $\epsilon \to 0$ this provides the equation Erwin Schrödinger wrote in 1925 which determines the time-evolution of the state vector as

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

for some hermitian operator H which is identified with the Hamiltonian of the particle from classical mechanics. The program of quantum mechanics involves studying solutions to this equation for various realizations of H. The eigenspectrum of H plays a crucial role in this theory.

Writing the Schrödinger equation in the position basis and separating variables $\Psi(t,x) = \phi(t)\psi(x)$ gives two ordinary differential equations

$$\frac{d\phi}{dt} = -\frac{i}{\hbar}E\phi \qquad\qquad H\psi = E\psi$$

The first of these is trivial to solve giving $\phi(t) = \exp(-iEt/\hbar)$. The second is an eigenvalue equation for the eigenspectrum of H. In many cases the eigenvalues are labelled by a discrete index E_n and the eigenstates as $|n\rangle$. After the entire eigenspectrum of H is known, the Schrödinger equation is solved by decomposing the initial state into this eigenbasis and applying time evolution to each[1]

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} e^{-\frac{iE_n}{\hbar}t}\langle n|\psi_0\rangle|n\rangle$$

This Hamiltonian formulation works in many cases. However in classical mechanics it is known that an equivalent Lagrangian formulation exists. This Lagrangian formulation of quantum mechanics is supplied by Feynman's path integral. Classically we would formulate the Lagrangian by considering a tra-

[1] Parts of this section are based on notes prepared by [8]

jectory $\mathbf{x}(t)$ between an initial point $(\mathbf{x}_i, t_i)$ and possible future point $(\mathbf{x}_f, t_f)$. We then associate to this trajectory a functional called its action, $S[\mathbf{x}(t)]$. The classical trajectory of the particle is then given as the trajectory $\mathbf{x}(t)$ which extremizes the action. In quantum mechanics, because of the uncertainty principle, we cannot speak of the particle as taking any definite path. Instead we can speak only of the probability that a particle transitioned from one state, $|\psi(\mathbf{x}_i, t_i)\rangle$, to another state $|\psi(\mathbf{x}_f, t_f)\rangle$. We define the transition probability amplitude as the inner product of the wavefunction evaluated at these two points. This quantity is known as the propagator

$$U(x_f, t_f; x_i, t_i) \equiv \langle \psi(\mathbf{x}_f, t_f) | \psi(\mathbf{x}_i, t_i) \rangle$$

In practice the propagator is difficult to calculate. The postulate, first given by Feynman, is to assume the contribution to the propagator from a particular trajectory is $\exp[iS[\mathbf{x}(t)]/\hbar]$. That is, every possible path contributes with equal amplitude to the propagator, but with a phase related to the classical action. For a given initial point and future point, summing over all possible trajectories gives the propagator (the normalization constant $A(t)$ is independent of any individual path and therefore depends only on time):

$$U(\mathbf{x}_f, t_f; \mathbf{x}_i, t_i) = A(t) \sum_{\text{all paths } \mathbf{x} \text{ from } \mathbf{x}_i \text{ to } \mathbf{x}_f} \exp\left[\frac{i}{\hbar} S[\mathbf{x}(t)]\right]$$

We integrate over all paths because we cannot speak of the particle as have taken any particular path. This propagator, together with the initial state, fully determines the evolution of the system. The propagator acts on a wave function ψ to propagate it forward in time by

$$\langle \mathbf{x}_f | \psi(t_f) \rangle = \int d\mathbf{x}_i \langle \psi(t_f, \mathbf{x}_f) | \psi(t_i, \mathbf{x}_i) \rangle | \psi(t_i, \mathbf{x}_i) \rangle$$

That is we also need to integrate over all possible initial points to determine the time evolution because we cannot say with certainty where the particle was originally located. This result, known as the path integral formulation, can be shown to reduce to the Schrödinger equation.

Quantum Field Theory[2]

The fundamental quantities of quantum field theory are the fields, ϕ. Particles are viewed as excitations of this field and interactions between particles are viewed as interactions between fields. These fields exists on a manifold M and are defined as maps between manifolds, that is $\phi : M \to X$ are fields on M. For example $X = \mathbb{R}$ describes a scalar field. The goal of quantum field theory is to determine the "equations of motion" of the fields. These can be obtained in precisely the analogous way as quantum mechanics by writing a Lagrangian as a function of the field ϕ and its derivatives $\partial_\mu \phi$ and solving the Euler-Lagrange equations on M. Note that because M can be a general manifold these will almost always make explicit reference to a metric $g_{\mu\nu}$. For a free massless scalar field the Lagrangian (density) is $\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$. We can add a potential term in the normal way as $\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi)$ where we have contracted the metric to define the partial with a raised index. The action functional is defined as

$$S[\phi] = \int_M \mathcal{L}(\phi, \partial_\mu \phi) \sqrt{\det(g)} d^n x$$

where $\sqrt{\det(g)} d^n x$ is the volume form on M. By analogy, the path integral is

$$\mathcal{Z}(M) = \int \mathcal{D}\phi e^{-S[\phi]}$$

where the integral is over all possible ϕ. Though this can only be given precise meaning in a handful of cases, it provides the foundation for all physics.

As we argued above, $\mathcal{Z}(M)$ is the time-evolution operator on the fields and, in principle, determines the evolution of the system. However in practice it is still necessary to find a way to describe the final and initial states in this language. What we want is to illustrate how the path integral can be used to describe particle interactions. What we need to do to is couple a spacetime dependent background field J to the Lagrangian which serves to create or destroy particles. We imagine this background field turning on and

[2] This section borrows from [2].

off as particles interact and annihilate. We can then expand the exponential in the path integral in powers of this field and recover an integral of the form

$$\langle O_1, \ldots, O_n \rangle_g = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi O_1 \cdots O_n e^{-S[\phi]}$$

where g indicates the metric on M. These integrals are almost never calculated directly but can instead be expressed in terms of Feynman diagrams. The variables $O_1, \ldots, O_n$ are called *observables of the field*. An observable is, by definition, a function from the set of field configurations $\{\phi : M \to X\}$ to the complex numbers. The path integral of the observables is called the *correlation function* of the observables. In general the correlation function will depend on the metric of spacetime because we coupled a spacetime-dependent background field to the Lagrangian to get it.

An interesting special case occurs when the correlation functions are independent of g. Because metric independence implies diffeomorphism invariance we have the interesting result that all isometries $f : M \to M'$ leave the path integral invariant. Stated more plainly, the theory is not sensitive to changes in the shape of the manifold.[3] But this is exactly what is meant by a topological field theory: it is a field theory which depends only on the topology.

What does this mean for computing the correlation functions? When the field theory is topological we know that the answer can only depend on the topological invariants of the manifold. Therefore we expect there should be a simpler way to solve the field theory by studying only the manifold topology. This is what leads to the saying that a topological quantum field theory is a quantum field theory which computes topological invariants.

We are almost prepared to state Atiyah's axioms for a topological quantum field theory and make precise the connection between topology and quantum field theory. Before we get there though we need to take a quick detour through category theory. Half of the battle of understanding Atiyah's axioms is in understanding the language in which they are stated.

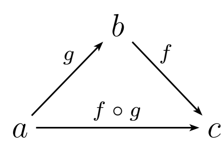
[3] If the manifold represents spacetime then the correlation functions do not change when the spacetime warps or contracts. This invariance is referred to as *background independence* and is viewed by some as an important step towards a quantum theory of gravity.

Category Theory[4]

The basic notion in category theory is that of a category, sometimes colloquially referred to as a 'mathematical universe'. A category, $\mathcal{C}$, consists of

Objects:	$a, b, c \ldots$
Maps:	$a \xrightarrow{f} b, \ldots$
Identity Maps:	$a \xrightarrow{1_a} a, \ldots$
Composition of maps:	Assigns to each pair of maps of the type $a \xrightarrow{g} b \xrightarrow{f} c$, another map $a \xrightarrow{f \circ g} c$

Other words commonly used for maps are 'morphism' and 'arrow'. Sometimes the set of objects of a category are denoted $\mathrm{ob}(\mathcal{C})$ and the set of arrows as $\mathrm{hom}(\mathcal{C})$ but more often they are both referred to simply by the name of the category, $\mathcal{C}$. We may describe the composition operation by saying that the following diagram commutes



There are many categories, each appropriate to a particular subject matter. Several categories include:

Set, the category of finite sets and maps
Grp, the category of groups and homomorphism
Ab, the category of abelian groups and homomorphisms
Rng, the category of rings and ring homomorphisms
$\mathrm{Vect}_{\mathbb{K}}$, the category of vector spaces over a field $\mathbb{K}$ and linear maps
Top, the category of topological spaces and continuous maps
Bord_n, the category of bordisms

[4]See [5] for a simple introduction to category theory. The standard text on category theory is [7]. This section quotes freely from both sources.

A quick check confirms that each of these categories contains the necessary data. To form a category we require that the identity and composition operations satisfy two axioms.

1. **Identity Laws**

   (a) If $a \xrightarrow{1_a} a \xrightarrow{f} b$

   then $a \xrightarrow{f \circ 1_a = f} b$

   (b) If $a \xrightarrow{f} b \xrightarrow{1_b} b$

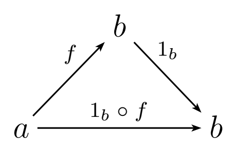
   then $a \xrightarrow{1_b \circ f = f} b$

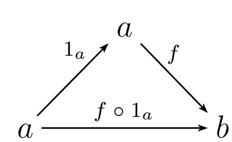
2. **Associativity:**

   If $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$

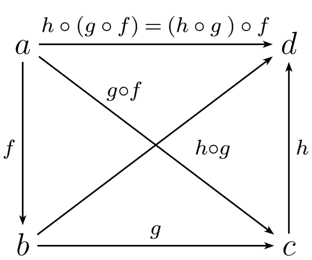
   then $a \xrightarrow{h \circ (g \circ f) = (h \circ g) \circ f} d$

The identity laws are equivalent to requiring the following two diagrams to commute





The associative law likewise has a commutative diagram



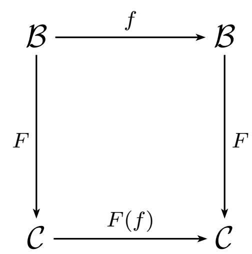
The framework identified so far already has much to say about many fields of mathematics. Take for example the category FinSet. The objects of FinSet are finite sets and the arrows are maps between sets. In set theory we already had the notion of equality of sets. This was inherited from the notion of equality of elements of a set. But set theory lacks a similar structure for identifying equality among the many maps of sets. Consider for example that there are many interesting maps between sets but there is a special class of maps which map between the same set. These maps are called *endomaps*. One of these endomaps is the identity, but there are many others. The identity is distinguished by sending every element to itself. Some of these endomaps are distinguished by pairing off elements and sending these pairs to each other (if the order of the set is odd at least one element has to be paired with itself). In this case the endomap is its own inverse indeed endomap are the only maps of sets which may be their own inverses. Such an endomap that is its own inverse is called an *involution*. Other maps pair elements off in threes or fours. Some endomaps pair some elements off in threes and other elements in pairs and yet others in more complicated ways. Intuitively we recognize that maps which have the same cycle structure are equivalent in the sense that they can be mapped onto each other while maintaining the relationships of their elements. This intuition is what category theory formalizes.

In category theory the previous observation would be recorded by giving the following commutative diagram in which α and β are two endomaps of the sets A and B (possibly the same set) and f is a third map between the two sets (itself possibly an endomap in the case A and B are the same)

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ B & \xrightarrow{\beta} & B \end{array}$$

The important lesson is that category theory provides us with the language to describe maps on the same footing as the objects themselves.

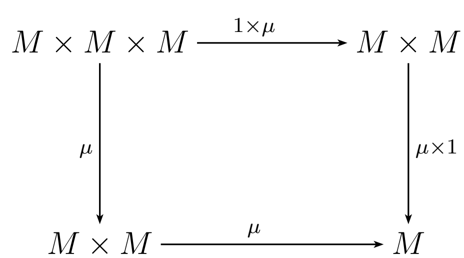
In addition categories there are ways to pass from one category to another. This gives the second most important feature of category theory – the functor. A functor is a morphism of categories and must map *both* objects and morphisms from one category to the other. That is, for two categories $\mathcal{B}$ and $\mathcal{C}$ a functor $F : \mathcal{B} \to \mathcal{C}$ consists: (1) the *object function* F, which assigns to each object b of $\mathcal{B}$ an object $F(b)$ of $\mathcal{C}$ and (2) the *morphism function* (also written F) which assigns to each morphism $b \xrightarrow{f} b'$ of $\mathcal{B}$ a morphism $F(b) \xrightarrow{F(f)} F(b')$ of $\mathcal{C}$. Note that the functor assigns the morphism of $\mathcal{B}$ not to just any morphism of $\mathcal{C}$ but to that morphism which is between the objects of $\mathcal{C}$ mapped to by those in $\mathcal{B}$. This relationship may also be given a commutative diagram.



This suggests that the category of categories is itself a category with morphisms given by functors. By providing a precise mechanism for translation of structure from one ‘mathematical universe’ to another, the concept of functor has allowed a new unification of many fields of mathematics.

We now define a refinement of the notion of category and functor: that of a monoidal category and a monoidal functor. A category M is called *monoidal* if, in addition to the data and rules already discussed, it contains a bifunctor $\mu : M \times M \to M$ called the monoidal product (sometimes tensor product). This monoidal product is useful because it allows us not only to compose morphisms in M but also objects. The first example of a monoidal category is $\mathrm{Vect}_{\mathbb{K}}$ where the ordinary tensor product $\otimes$ composes both vector spaces and maps on those vector spaces.

We require μ to be associative up to natural isomorphism and to have an object which is an identity. We can write this object as the map $\eta : 1 \to M$. These rules are summarized by the two commutative diagrams[5]



and

[5] These two diagrams and the following two are often superseded by the so-called pentagon diagram and triangle diagram which are an equivalent set of commutative diagrams which define a monoidal category.

$$\begin{array}{ccccc} 1\times M & \xrightarrow{\eta\times 1} & M\times M & \xrightarrow{1\times\eta} & M\times 1 \\ \Big\downarrow{\scriptstyle\lambda} & & \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\rho} \\ M & = & M & = & M \end{array}$$

Where λ and ρ are maps $1 \times X \xrightarrow{\lambda} X \xleftarrow{\rho} X \times 1$ given by 'projection': $\lambda(0, x) = x = \rho(x, 0)$.

The above defining diagrams for a monoidal category may be clarified by rewriting with elements. We write μ as a product $\mu(x, y) = xy$ and replace the function η on the 1-point set 1=$\{0\}$ by its only value, an element $\eta(0) = u \in M$.

$$\begin{array}{ccc} \langle x, y, z\rangle & \longmapsto & \langle x, yz\rangle \\ \Big\downarrow & & \Big\downarrow \\ \langle xy, z\rangle & \longmapsto & (xy)z = x(yz) \end{array}$$

$$\begin{array}{ccccccc} \langle 0, x\rangle & \longmapsto & \langle u, x\rangle & & \langle x, u\rangle & \longleftarrow\!\shortmid & \langle x, 0\rangle \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ x & = & ux & & xu & = & x \end{array}$$

These are precisely the familiar axioms of a monoid: multiplication is associative and there is an element u that is a left and right identity. Hence we have a monoidal category. A monoidal *functor* (sometimes called a tensor functor) is a functor $\mathcal{Z}$ which maps between two monoidal categories and respects the monoidal product. If $\mathcal{Z}$ is a monoidal functor between categories with monoidal products denoted by $\sqcup$ and $\otimes$ and if Σ and Σ' are two objects in the first category then $\mathcal{Z}$ is required to satisfy

$$\mathcal{Z}(\Sigma \sqcup \Sigma') = \mathcal{Z}(\Sigma) \otimes \mathcal{Z}(\Sigma')$$

One consequence of this requirement is that $\mathcal{Z}$ should take the tensor unit of the first category to the tensor unit of the second. For example, in $\text{Vect}_{\mathbb{K}}$ the unit of the tensor product is the ground field: $\mathbb{K} \otimes V \cong V \cong V \otimes \mathbb{K}$. If in some other category the unit of the monoidal product is denoted $\varnothing$ then a monoidal functor from that category to $\text{Vect}_{\mathbb{K}}$ is required to satisfy

$$\mathcal{Z}(\Sigma) = \mathcal{Z}(\varnothing \sqcup \Sigma) = \mathcal{Z}(\varnothing) \otimes \mathcal{Z}(\Sigma)$$

So clearly a monoidal functor must satisfy $\mathcal{Z}(\varnothing) = \mathbb{K}$.

Lastly, a monoidal category is called symmetric if there exists a natural isomorphism

$$\gamma : x \otimes y \to y \otimes x$$

called the *braiding* which satisfies the condition

$$\gamma \circ \gamma = 1$$

for all objects x, y. The braiding is sometimes referred to as a ‘twist’ and expresses a certain commutativity in the monoidal product (here written as $\otimes$). In general a monoidal category may be braided without being symmetric. What this means is that there may exist a natural isomorphism when the order of products is reversed but that reversing the order twice does not necessarily equal the original product (the twist is not necessarily its own inverse).

Summarizing, a symmetric monoidal functor is a functor $\mathcal{Z} : \mathcal{B} \to \mathcal{C}$ between symmetric monoidal categories is a monoidal functor which respects the symmetry of both categories. This is expressed in the following commutative diagram.

$$\begin{array}{ccc} a \sqcup b & \xrightarrow{\gamma} & b \sqcup a \\ \downarrow{\scriptstyle \mathcal{Z}} & & \downarrow{\scriptstyle \mathcal{Z}} \\ \mathcal{Z}(a) \otimes \mathcal{Z}(b) & \xrightarrow{\mathcal{Z}(\gamma)} & \mathcal{Z}(b) \otimes \mathcal{Z}(a) \end{array}$$

The axioms of topological quantum field theory

With these definition we are prepared to explain Atiyah's definition of a topological quantum field theory.

Definition (Atiyah): *An n-dimensional topological quantum field theory is a symmetric monoidal functor*

$$\mathcal{Z} : \mathrm{Bord}_n \to \mathrm{Vect}_{\mathbb{C}}$$

We will work through this definition slowly. The category $\mathrm{Vect}_{\mathbb{C}}$ is already familiar as the prototypical monoidal category. As a category it consists of

Objects:	Vector spaces over $\mathbb{C} : A, B, C \ldots$
Maps:	Linear maps $A \xrightarrow{f} A, A \xrightarrow{f} B \ldots$
Identity Maps:	$A \xrightarrow{1_A} A, \ldots$
Composition of maps:	Assigns to each pair of maps of the type $A \xrightarrow{g} B \xrightarrow{f} C,$ another map $A \xrightarrow{f \circ g} C$
Monoidal Product:	(Tensor Product) $A \otimes B \in \mathrm{Vect}_{\mathbb{C}}$
	Satisfying $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$
	With unit $\mathbb{C}$: $\mathbb{C} \otimes A \cong A \cong A \otimes \mathbb{C}$
Symmetry:	$A \otimes B \cong B \otimes A$

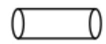
Bord_n is another monoidal category, perhaps less familiar. As a category it consists of (the less familiar objects are explained below)

Objects:	Closed $(n-1)$-manifolds: $\Sigma, \Sigma', \ldots$
Maps:	Bordisms, modulo diffeomorphism: $M, N, \ldots$
Identity Maps:	Cylinders
Composition of maps:	Gluing of bordisms
Monoidal Product:	(Disjoint Union) $\Sigma \sqcup \Sigma' \in \text{Bord}_n$
	Satisfying $(\Sigma \sqcup \Sigma') \sqcup \Sigma'' \cong \Sigma \sqcup (\Sigma' \sqcup \Sigma'')$
	With the empty manifold as unit :
	$\varnothing \sqcup \Sigma \cong \Sigma \cong \Sigma \sqcup \varnothing$
Symmetry:	$\Sigma \sqcup \Sigma' \cong \Sigma' \sqcup \Sigma$

The category Bord_n is named after its morphisms. A morphism $\Sigma \to \Sigma'$ is an equivalence class of bordisms from Σ to Σ'. A bordism $\Sigma \to \Sigma'$ is an oriented compact n-manifold M with boundary. We imagine that the boundary is the two manifolds Σ, Σ'. This is already rather abstract and manifolds with $n > 2$ are hard to visualize so we provide an example now of a 2-dimensional TQFT. This discussion will mirror that given by [6].

For $n = 2$ we need to consider the category Bord_2. The objects of Bord_2 are closed 1-manifolds. The only closed 1-manifold is the circle S^1. Because Bord_2 is a monoidal category with monoidal product given by disjoint union we can compose this object to get all the other objects. For example $S^1 \sqcup S^1$ is also an object of Bord_2. The functor $\mathcal{Z}$ is supposed to map objects from Bord_2 to vector spaces. Thus we have $\mathcal{Z}(S^1) = A$ for some vector space A. Because $\mathcal{Z}$ is a tensor functor we have that $\mathcal{Z}(S^1 \sqcup S^1) = A \otimes A$. This shows us that the entire action of $\mathcal{Z}$ for *all* objects of Bord_2 is determined by what it does to the circle. This is a special feature of TQFT in dimension 2 that it is determined by its action on a single object.

Morphisms in Bord_2 are compact 2-manifolds with boundary such that the boundary is some disjoint union of circles. First we have the identity bordism that is the cylinder $S^1 \times [0,1]$[6]



Sending this bordism through the functor gives $\mathcal{Z}(S^1 \to S^1) = A \to A$. This corresponds to no evolution. For a slightly less trivial example we have the bordism $B : S^1 \sqcup S^1 \to S^1$ from two circles to one circle (the pair of pants)

By sending B through $\mathcal{Z}$ this bordism corresponds to the map

$$\mathcal{Z}(B) = m : A \otimes A \to A$$

We can think of this as multiplication on A, as it is associative, commutative, and has a unit element. There is also the coproduct map from one circle to two circles

The unit comes from the bordism from the empty manifold to the circle

with map $Z(\varnothing \to S^1) : \mathbb{C} \to A$. This map is defined by where it sends 1. Likewise we have the bordism from the circle to the empty manifold

[6]To save time, instead of drawing my own bordisms I have copied those from [4].

with map $Z(S^1 \to \varnothing) = tr : A \to \mathbb{C}$ which we call the trace.
The last bordism we will explicitly define is the twist from two circles to two circles

This guarantees that the categories are symmetric. It turns out to be true that these six bordisms under composition and disjoint union generate the category Bord_2. To be explicit about this we could give a series of relations for these generators which we will not do here but only remark that they are equivalent to the Reidemeister moves from knot theory with the exception of not representing moves as passing over or under and that Witten used 3-d TQFT to make statements about the Jones polynomial and knot invariants.

The physical interpretation is as follows. One can imagine the $(n-1)$-manifolds Σ as space and that $\mathcal{Z}$ associates to this space a vector space that is the state space $\mathcal{H}_\Sigma$. The bordisms then represent spacetime and $\mathcal{Z}$ of the bordism gives a linear map/ operator that is identified with the Feynman propagator of the state space. That is, for each oriented n-manifold M with boundary $\partial M = \Sigma$, we obtain a vector $\mathcal{Z}(M) \in \mathcal{H}_\Sigma$ which corresponds to the propagator of the path integral. To form this propagator for fields ϕ on Σ we only need to calculate the path integral over fields on M which restrict to ϕ on the boundary:

$$\mathcal{Z}(M)(\phi) = \int_{\Phi \text{ on } M \text{ s.t. } \Phi|_\Sigma = \phi} \mathcal{D}\Phi e^{-S[\Phi]}$$

Diffeomorphism invariance of the bordisms ensures metric independence of correlation functions. Finally it was important that $\mathcal{Z}$ be a monoidal functor: that disjoint union goes to tensor product captures the idea that two systems which carry their own degrees of freedom correspond to tensored state spaces. In particular we have that $\mathcal{Z}(\varnothing) = \mathbb{C}$. Lastly, $\mathcal{Z}$, maps unit morphisms to unit morphisms. That is, for an oriented $(n-1)$-manifold Σ, we have $\mathcal{Z}(\Sigma \times [0,1]) = 1_{\mathcal{Z}(\Sigma)}$. This states the interesting fact that nothing happens in empty space, or that the Hamiltonian is zero.

We remark that composing these bordisms gives the nondegenarate trace pairing $A \otimes A \xrightarrow{m} A \xrightarrow{tr} \mathbb{C}$ represented by the bordism

This is the basis for the identification of 2-d TQFTs with commutative Frobenius algebras. To see this connection recall the definition of a $\mathbb{C}$-algebra as a $\mathbb{C}$-vector space A together with two $\mathbb{C}$-linear maps

$$m : A \otimes A \to A \qquad \eta : \mathbb{C} \to A$$

These maps must satisfy the associativity and unit laws given previously. This of course identifies the $\mathbb{C}$-algebra with a monoid in $\text{Vect}_{\mathbb{C}}$ and therefore also with Bord_2. The nondegenerate trace pairing given by the U-tube above makes this into a Frobenius algebra and the twist map makes it commutative.

In the theory of TQFTs closed n-manifolds without boundary play important roles. Such a manifold represents a bordism from the empty (n - 1)-manifold to itself, and its image under A is therefore a linear map $\mathbb{C} \to \mathbb{C}$ (a scalar). This scalar is a topological invariant in the sense that it is uniquely defined by the topology of the chosen closed n-manifold. This is the invariant we want to calculate. The strategy for an arbitrary closed n-manifold is to cut it into smaller pieces for which this invariant is easier to calculate and then paste the bordisms back together. This is especially evident in 2-dimensional TQFTs. Bord_n for $n > 2$ is hard to describe but in $n = 2$ not only is only one closed, connected 1-manifold, but in addition all closed 2-manifolds are known. They are given of course by the sphere, the torus, the double torus, and so forth. These manifolds are differentiated by their genus (number of holes) and Euler characteristic (a function of the number of holes).

Dijkgraaf-Witten Theory

The discussion in this section derives from [1] and [3].

Physically Dijkgraaf-Witten theory is recognized as G-gauge theory. The simplest example of a G-gauge theory is given by the Ising model of statistical mechanics used to model magnets. Magnetism is the result of the alignment of a macroscopically large number of electron spins which may either point up or down. In the Ising model the electrons are approximated as living on a lattice in d dimensions and each of the spins may be either up or down. Whether the spin is up or down corresponds to a weight in the Boltzmann partition function of the system defined in statistical mechanics. The dynamics of the systems are determined by this partition function.

We may view the Ising model as an example of a G-gauge theory where $G = \mathbb{Z}/2\mathbb{Z}$ as the spins can choose out of two orientations. We see then what is meant when Dijkgraaf-Witten theory is a G-gauge theory. Instead of being on a lattice we allow the theory on a manifold M and the 'choices' of spin orientation are identified with principle bundles for the group.[7] The connection to field theory is that the fields on M are these principle bundles. Moreover these are characterized by monodromy (essentially genus) on M. This connection identifies principle bundles with the fundamental group of M as $\{\pi_1(M) \to G\}$/conjugacy. This is the hint that the solution is topological despite that in Dijkgraaf-Witten theory the spins are at first put onto the manifold by triangulation of the manifold. The theory must consist of a map that sends combinations of the group elements to a Boltzmann weight which leads to the partition function. It can be shown that this partition function is independent of the original triangulation and thus Dijkgraaf-Witten theory is topological.

We want to reduce to the problem to knowing $Z(S^1)$ and then cut and

[7] We point out that Dijkgraaf-Witten theory is a toy model for Chern-Simons theory in which the group is finite.

paste the closed manifold into manageable ones. This cutting and pasting is done on 2-manifolds by triangulation. This is explained in [3]. A triangulation of M allows the definition of trace maps which form the invariants.

> Fields on M are homomorphisms from f.g. to G so they are given by $\frac{G}{G}$ adjoint quotient.
>
> Linearization of space of fields gives $Z(S^1) = A = Fun\left(\frac{G}{G}\right) = \mathbb{C}\left[\frac{G}{G}\right] = Z(\mathbb{C}[G])$ the space of class functions. Conjugacy classes (Characters of reps) of G are basis of A Hilbert space. Frobenius form given by trace (trace at 1 is dimension of a character)
>
> Simultaneously diagonalize operators in $A = \mathbb{C}\left[\frac{G}{G}\right] \leftrightarrow$ spectral decomposition
>
> Spec $\mathbb{C}\left[\frac{G}{G}\right] = \left\{ \text{ homomorphisms } \mathbb{C}\left[\frac{G}{G}\right] \to \mathbb{C}\right\} = \left\{ \text{ Irreps of } G\right\}$

This result on the invariant of the TQFT can be used to derive Mednykh's formula which places constraints on the dimensions of irreducible representations of the group G given characteristics of the manifold M.

$$\sum_{V \in \text{ Irrep}(G)} \text{dim} V^{\chi(M)} = |G|^{\chi(M)-1} |\text{Hom}(\pi_1(M), G)|$$

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